TRIANGULARISATION METHOD

It is also known as Doolittle Method and Factorization Method. Therefore we factorize the given matrix A as LU i.e., $A = LU$ ——— (1)

where 'L' is a lower triangular matrix having diagonal elements unity and 'U' is an upper triangular matrix. i.e.,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Now, from (1), we get

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} \quad \text{——— (2)}$$

Determination of L^{-1} – To find L^{-1} let us take $L^{-1} = X$, where 'X' is also a lower triangular matrix

$\therefore LX = I$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the matrices on the R.H.S. and equating of both sides of matrices we get

$x_{11} = 1$; $x_{22} = 1$; $x_{33} = 1$

$l_{21}x_{11} + x_{21} = 0$;

$l_{31}x_{11} + l_{32}x_{21} + x_{31} = 0$;

$l_{32}x_{22} + x_{32} = 0$

From above equations, we get all the elements of X and hence, we obtain $L^{-1} = X$.

Determination of U^{-1}: To find U^{-1}, let us take $U^{-1} = Y$, where Y is also an upper triangular matrix

i.e., $UY = I$

$$\therefore \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, multiplying the matrices on R.H.S. and equating of both sides the corresponding elements of the matrices, we get

$U_{11} y_{11} = 1$; $U_{22} y_{22} = 1$; $U_{33} y_{33} = 1$

$U_{11} y_{12} + U_{12} y_{22} = 0$;

$U_{11} y_{13} + U_{12} y_{23} + U_{13} y_{33} = 0$

$U_{22} y_{23} + U_{23} y_{33} = 0$

From these equations, we get all athe elements of Y and hence, we obtain $U^{-1} = Y$

Consequently from ②, we obtain A^{-1}.

Ques) Using the triangularisation method, find the inverse of the matrix

$$A = \begin{bmatrix} 50 & 107 & 36 \\ 25 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix}$$

Soln) Let $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} ; \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\therefore \begin{bmatrix} 50 & 107 & 36 \\ 25 & 54 & 20 \\ 31 & 66 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore \quad u_{11} = 50 \; ; \; u_{12} = 107 \; ; \; u_{13} = 36$

$l_{21} u_{11} = 25 \Rightarrow l_{21} = 25/50 = 1/2$

$l_{21} u_{12} + u_{22} = 54$

$\Rightarrow u_{22} = 54 - l_{21} u_{12}$

$= 54 - \frac{107}{2} = 1/2$

$l_{21} u_{13} + u_{23} = 20$

$\Rightarrow u_{23} = 20 - l_{21} u_{13}$

$= 20 - 18 = 2$

$l_{31} u_{11} = 31 \Rightarrow l_{31} = 31/50$

$l_{31} u_{12} + l_{32} u_{22} = 66$

$\Rightarrow l_{32} = \frac{1}{u_{22}} (66 - l_{31} u_{12})$

$= 2\left(66 - \frac{31}{50} \cdot 107\right) = 132 - \frac{3317}{25}$

$= -(17/25)$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 21$

$\Rightarrow u_{33} = 21 - l_{31} u_{13} - l_{32} u_{23}$

$= 21 - \frac{31}{50} \cdot 36 + \frac{17}{25} \cdot 2 = 21 - \frac{558}{25} + \frac{34}{25}$

$= (1/25)$

Thus, $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 31/50 & -17/25 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 50 & 107 & 36 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/25 \end{bmatrix}$

Determination of L^{-1} and U^{-1} :

Let $L^{-1} = X \Rightarrow LX = I$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 31/50 & -17/25 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore x_{11} = 1 \; ; x_{22} = 1 \; ; x_{33} = 1 \; ;$

$\frac{1}{2}x_{11} + x_{21} = 0 \Rightarrow x_{21} = -\frac{1}{2}x_{11} \Rightarrow x_{21} = (-1/2)$

$\frac{31}{50}x_{11} - \frac{17}{25}x_{21} + x_{31} = 0 \Rightarrow x_{31} = \frac{17}{25}x_{21} - \frac{31}{50}x_{11}$

$\Rightarrow x_{31} = \frac{17}{25}\left(\frac{-1}{2}\right) - \frac{31}{50}(1) = -24/25$

$-\frac{17}{25}x_{22} + x_{32} = 0 \Rightarrow x_{32} = \frac{17}{25}x_{22} = \frac{17}{25}$

$$\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -24/25 & 17/25 & 1 \end{bmatrix}$$

• Now, let $U^{-1} = Y \Rightarrow UY = I$

$$\begin{bmatrix} 50 & 107 & 36 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/25 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore 50y_{11} = 1 \Rightarrow y_{11} = 1/50 \; ; (1/2)y_{22} = 1 \Rightarrow y_{22} = 2$

$\frac{1}{25}y_{33} = 1 \Rightarrow y_{33} = 25$

$50y_{12} + 107y_{22} = 0 \Rightarrow y_{12} = -\frac{107}{50}y_{22} = -\frac{107}{25}$

$\frac{1}{2}y_{23} + 2y_{33} = 0 \Rightarrow y_{23} = -4y_{33} = -100$

$50y_{13} + 107y_{23} + 36y_{33} = 0$

$\Rightarrow y_{13} = \frac{1}{50}(-107y_{23} - 36y_{33}) = \frac{1}{50}(-107(-100) - 35(25))$

$= \frac{1}{50}(10700 - 900) = 196$

$$\text{Thus } U^{-1} = Y = \begin{bmatrix} 1/50 & -107/25 & 196 \\ 0 & 2 & -100 \\ 0 & 0 & 25 \end{bmatrix}$$

$$\text{Hence } A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/50 & -107/25 & 196 \\ 0 & 2 & -100 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -24/25 & 17/25 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -186 & 129 & 196 \\ 95 & -66 & -100 \\ 24 & 17 & 25 \end{bmatrix} \text{ Ans}$$